METHODOLOGY OF MATRIX REPRESENTATION OF HIGHER ORDER TENSORS

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1. INTRODUCTION

With higher order tensors (four, six, eight, etc.) we meet in the study of relations between stress and strain. In the case of reversible processes constitutive equations are written in the form

$$t_{ij} = c_{ijnm} d_{nm} + c_{ijnmpq} d_{nm} d_{pq} + c_{ijnmpqkl} d_{nm} d_{pq} d_{kl} + ... \qquad (1)$$

where - t_{ij}, d_{ij} are denoted stress strain tensors respectively, and by - c_{ijnm}, c_{ijnmpq}, $c_{ijnmpqkl}$ - the elasticity constants tensors of the order of the fourth, sixth and eighth.

From symmetry of stress, strain tensors and the laws of thermodynamics, for tensors of elasticity constants resulting the following symmetry relations

$$c_{ijnm} = c_{jinm} = c_{ijmn} = c_{nmij}, \qquad (2)$$

$$c_{ijnmpq} = c_{jinmpq} = c_{ijmnpq} = c_{ijmnqp} = c_{nmijpq} =$$
$$= c_{pqnmij} = c_{ijpqnm} \qquad (3)$$

$$c_{ijnmpqkl} = c_{jinmpqkl} = c_{ijmnpqkl} = c_{ijnmqpkl} =$$
$$= c_{ijnmpqlk} = c_{nmijpqkl} = c_{k\ln mpqij}. \qquad (4)$$

Depending on the type of interactions among atoms or molecules, the relations (2), (4) the additional information can be added.

If, for example, interactions between atoms or molecules are central (ionic bonding), the elastic constant tensors of any order is totally symmetrical. Recall that a tensor is totally symmetrical if it is symmetric in relation to all pairs of indices. In the case of the fourth order tensors the relation takes place

$$c_{ijnm} = c_{injm}. \qquad (5)$$

The material symmetry which is expressed quantitatively by the planes of symmetry and symmetry axes of different order, leads to a reduction in the number of independent constants of elasticity.

2. MATRIX REPRESENTATION OF FOURTH ORDER TENSOR

The experimental data for components of elasticity constants tensors shown in the crystallographic coordinate system, containing only independent sizes.

Calculation of elastic constants in an arbitrary coordinate system is simplified considerably if higher order tensors are represented by composed matrix [1]. The fourth order tensor can be presented in the form of

$$c_{ijnm} = (c_{ij})_{nm}, \qquad (6)$$

where - $(c_{ij})_{nm}$ is a square composed matrix of the second order, each element of which is also a square matrix, i.e.

$$C := \begin{bmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{11} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{12} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{13} \\ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{21} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{22} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{23} \\ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{31} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{32} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{33} \end{bmatrix} \qquad (7)$$

For the fourth order tensor which has the symmetry properties (2) the components of composed matrix are expressed only by 21 independent constants. The 21 independent constants can be presented as a 21x1 column matrix, the elements which we will denote by a_I, where I=1,2,...,21.

Thus, the tensor of elasticity constants must be expressed as follows

$$C := \begin{bmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{pmatrix} & \begin{pmatrix} a_2 & a_7 & a_8 \\ a_7 & a_9 & a_{10} \\ a_8 & a_{18} & a_{11} \end{pmatrix} & \begin{pmatrix} a_3 & a_8 & a_{12} \\ a_8 & a_{13} & a_{14} \\ a_{12} & a_{14} & a_{15} \end{pmatrix} \\ \begin{pmatrix} a_2 & a_7 & a_8 \\ a_7 & a_9 & a_{10} \\ a_8 & a_{18} & a_{11} \end{pmatrix} & \begin{pmatrix} a_4 & a_9 & a_{13} \\ a_9 & a_{16} & a_{17} \\ a_{13} & a_{17} & a_{18} \end{pmatrix} & \begin{pmatrix} a_5 & a_{10} & a_{14} \\ a_{10} & a_{17} & a_{19} \\ a_{14} & a_{19} & a_{20} \end{pmatrix} \\ \begin{pmatrix} a_3 & a_8 & a_{12} \\ a_8 & a_{13} & a_{14} \\ a_{12} & a_{14} & a_{15} \end{pmatrix} & \begin{pmatrix} a_5 & a_{10} & a_{14} \\ a_{10} & a_{17} & a_{19} \\ a_{14} & a_{19} & a_{20} \end{pmatrix} & \begin{pmatrix} a_6 & a_{11} & a_{15} \\ a_{11} & a_{18} & a_{20} \\ a_{15} & a_{20} & a_{21} \end{pmatrix} \end{bmatrix} \quad (8)$$

If the tensor is totally symmetrical relations may occur

$$a_8 = a_5, a_7 = a_4, a_{12} = a_6,$$
$$a_{19} = a_{18}, a_{13} = a_{10}, a_{14} = a_{11} \quad (9)$$

In the case of orthotropic materials the matrix of elasticity constants is expressed as

$$C := \begin{bmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_6 \end{pmatrix} & \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & a_{12} \\ 0 & 0 & 0 \\ a_{12} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} a_4 & 0 & 0 \\ 0 & a_{16} & 0 \\ 0 & 0 & a_{18} \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{19} \\ 0 & a_{19} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & a_{12} \\ 0 & 0 & 0 \\ a_{12} & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{19} \\ 0 & a_{19} & 0 \end{pmatrix} & \begin{pmatrix} a_6 & 0 & 0 \\ 0 & a_{18} & 0 \\ 0 & 0 & a_{21} \end{pmatrix} \end{bmatrix} \quad (10)$$

For materials with cubic symmetry (11).

The relationships between the stress and strain in an arbitrary coordinate system in the linear approximation is determined from the relation (12)

$$C := \begin{bmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_4 \end{pmatrix} & \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & a_7 \\ 0 & 0 & 0 \\ a_7 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} a_4 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_4 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_7 \\ 0 & a_7 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & a_7 \\ 0 & 0 & 0 \\ a_7 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_7 \\ 0 & a_7 & 0 \end{pmatrix} & \begin{pmatrix} a_4 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \end{bmatrix} \quad (11)$$

$$d_{in} = \sum_{k=1}^{3}\sum_{q=1}^{3}\left[\sum_{c=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{j=1}^{3}\left[r_{ij} r_{nm} r_{kl} r_{qc} (C_{jm})_{lc}\right] t_{kq}\right], (12)$$

where r_{ij} the matrix of rotation is denoted which is used to determine the position given by the coordinates system to the crystallographic system. Rotation matrix is obtained as a result of three successive rotations and are calculated according to the formula

$$\boldsymbol{r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & \sin(\theta_1) \\ 0 & -\sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_1) \end{pmatrix} \boldsymbol{x}$$

$$\boldsymbol{x} \begin{pmatrix} \cos(\theta_3) & \sin(\theta_3) & 0 \\ -\sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (13)$$

were $\theta_1, \theta_2, \theta_3$ - are Euler angles.

3. A MATRIX REPRESENTATION OF SIX ORDER TENSOR

In case of nonlinear relations between stress and strain the six and eight order tensors are intervened, and these tensors can be presented by composed matrix. In base of symmetric relations (2)-(4) is posible to pass from two indexs notations to one index after Viogt [2] convention

$$11 \sim 1, 22 \sim 2, 33 \sim 3, 23 \sim 4, 13 \sim 5, 12 \sim 6 .$$

Adopting this convention, we will write

$$c_{ijnm} = c_{KM} ,\; c_{ijmnrs} = c_{KMF}$$

$$c_{ijmnrskq} = c_{KMFL} ,$$

where the small letters have the values 1,2,3, but big 1,2,...,6. In plus, the symmetric relations (1) – (3) give

$$c_{KM} = c_{MK} ,\; c_{KMF} = c_{MKF} = c_{FMK} = c_{KFM}$$

$$c_{KMFL} = c_{MKFL} = c_{KMLF} = c_{MKLF} = c_{LFKM} =$$

$$= c_{LFMK} = c_{FLKM} = c_{FLMK} = c_{FKLM} = \ldots$$

Matrixes $c_{KM}, c_{KMF}, c_{KMFL}$ don’t represent the tensor in obtained meaning. Therefore, in rule of components transformation at rotation of reference system doesn’t directly given the rotation matrix *r*.

It can be demonstrated, that for these matrixes can be used the known rules of components transformation, so

$$c'_{KM} = R_{KI} R_{MJ} c_{IJ}$$

$$c'_{KMF} = R_{KI} R_{MG} R_{FT} c_{IGT}$$

$$c'_{KMFL} = R_{KI} R_{MG} R_{FT} R_{LU} c_{IGTU} ,$$

were *K,M,…,U=1,2,…,6.* *R* matrix is presented [3]

$$R = \begin{pmatrix} r_{11}^2 & r_{12}^2 & r_{13}^2 & r_{12}r_{13} & r_{13}r_{11} & r_{11}r_{12} \\ r_{21}^2 & r_{22}^2 & r_{23}^2 & r_{22}r_{23} & r_{23}r_{21} & r_{21}r_{22} \\ r_{31}^2 & r_{32}^2 & r_{33}^2 & r_{32}r_{33} & r_{33}r_{31} & r_{31}r_{32} \\ 2r_{21}r_{31} & 2r_{22}r_{32} & 2r_{23}r_{33} & r_{22}r_{33}+r_{23}r_{32} & r_{23}r_{31}+r_{21}r_{33} & r_{21}r_{32}+r_{22}r_{31} \\ 2r_{31}r_{11} & 2r_{32}r_{12} & 2r_{33}r_{13} & r_{32}r_{13}+r_{33}r_{12} & r_{33}r_{11}+r_{31}r_{13} & r_{31}r_{12}+r_{32}r_{11} \\ 2r_{11}r_{21} & 2r_{12}r_{22} & 2r_{13}r_{23} & r_{12}r_{23}+r_{13}r_{22} & r_{13}r_{21}+r_{11}r_{23} & r_{11}r_{22}+r_{12}r_{21} \end{pmatrix} \quad (14)$$

The matrix $(\boldsymbol{C_I})_{\boldsymbol{GT}}$ we can present

$$C' := \begin{bmatrix} \begin{bmatrix} \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_2 & c_7 & c_8 & c_9 & c_{10} & c_{11} \\ c_3 & c_8 & c_{12} & c_{13} & c_{14} & c_{15} \\ c_4 & c_9 & c_{13} & c_{16} & c_{17} & c_{18} \\ c_5 & c_{10} & c_{14} & c_{17} & c_{19} & c_{20} \\ c_6 & c_{11} & c_{15} & c_{18} & c_{20} & c_{21} \end{pmatrix} \\ \begin{pmatrix} c_2 & c_7 & c_8 & c_9 & c_{10} & c_{11} \\ c_7 & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_8 & c_{23} & c_{27} & c_{28} & c_{29} & c_{30} \\ c_9 & c_{27} & c_{28} & c_{31} & c_{32} & c_{33} \\ c_{10} & c_{25} & c_{29} & c_{32} & c_{34} & c_{35} \\ c_{11} & c_{16} & c_{30} & c_{33} & c_{35} & c_{36} \end{pmatrix} \\ \begin{pmatrix} c_3 & c_8 & c_{12} & c_{13} & c_{14} & c_{15} \\ c_8 & c_{23} & c_{27} & c_{28} & c_{29} & c_{30} \\ c_{12} & c_{13} & c_{37} & c_{38} & c_{39} & c_{40} \\ c_{13} & c_{28} & c_{38} & c_{41} & c_{42} & c_{43} \\ c_{14} & c_{29} & c_{39} & c_{42} & c_{44} & c_{45} \\ c_{15} & c_{30} & c_{40} & c_{43} & c_{45} & c_{46} \end{pmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{pmatrix} c_4 & c_9 & c_{13} & c_{16} & c_{17} & c_{18} \\ c_9 & c_{24} & c_{28} & c_{31} & c_{32} & c_{33} \\ c_{13} & c_{28} & c_{38} & c_{41} & c_{42} & c_{43} \\ c_{16} & c_{31} & c_{41} & c_{47} & c_{48} & c_{49} \\ c_{17} & c_{32} & c_{42} & c_{48} & c_{50} & c_{51} \\ c_{18} & c_{33} & c_{43} & c_{49} & c_{51} & c_{52} \end{pmatrix} \\ \begin{pmatrix} c_5 & c_{10} & c_{14} & c_{17} & c_{19} & c_{20} \\ c_{10} & c_{25} & c_{29} & c_{32} & c_{34} & c_{35} \\ c_{14} & c_{29} & c_{39} & c_{42} & c_{44} & c_{45} \\ c_{17} & c_{32} & c_{44} & c_{48} & c_{50} & c_{51} \\ c_{19} & c_{34} & c_{44} & c_{50} & c_{53} & c_{54} \\ c_{20} & c_{35} & c_{45} & c_{51} & c_{54} & c_{55} \end{pmatrix} \\ \begin{pmatrix} c_6 & c_{11} & c_{15} & c_{18} & c_{20} & c_{21} \\ c_{11} & c_{26} & c_{30} & c_{33} & c_{35} & c_{36} \\ c_{15} & c_{30} & c_{40} & c_{43} & c_{45} & c_{46} \\ c_{18} & c_{35} & c_{43} & c_{49} & c_{51} & c_{52} \\ c_{20} & c_{35} & c_{45} & c_{51} & c_{54} & c_{55} \\ c_{21} & c_{36} & c_{46} & c_{52} & c_{55} & c_{56} \end{pmatrix} \end{bmatrix} \end{bmatrix}$$

So, the tensor of elastic constants of fourth order is expressed by 21 independent components, but six order tensor by 56. These 56 components are presented by column matrix with 56x1 dimensions.

The number of independent constants of elasticity is reduced, if materials have and other elements of symmetry. For materials with cubic symmetry the number of elasticity constants of stress tensor of six order is decreased up to six.

The only non-zero constants of elasticity tensor of six order are

$$c_1 = C_{111} = c_{22} = C_{222} = c_{37} = C_{333,}$$

$$c_2 = C_{112} = c_3 = C_{113} = c_7 = C_{122} = c_{23} = C_{223} =$$
$$\boldsymbol{c}_{12} = \boldsymbol{C}_{133} = \boldsymbol{c}_{27} = \boldsymbol{C}_{233}\boldsymbol{,}$$

$$c_8 = C_{123}, c_{51} = C_{456},\ c_{16} = C_{144} = c_{34} = C_{255} =$$
$$\boldsymbol{c}_{46} = \boldsymbol{C}_{366}\boldsymbol{,}$$

$$c_{19} = C_{155} = c_{21} = C_{166} = c_{31} = C_{244} = c_{36} = C_{266} =$$
$$\boldsymbol{c}_{41} = \boldsymbol{C}_{344} = \boldsymbol{c}_{44} = \boldsymbol{C}_{355}.$$

Therefore, the elastic behavior of material of cubic symmetry in approximation

$$\boldsymbol{t_I} = \boldsymbol{C_{IN} d_N} + \boldsymbol{(C'_I)_{NM} d_N d_M}$$

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_4 & \boldsymbol{a}_6 & \boldsymbol{a}_5 & \boldsymbol{a}_3 & \boldsymbol{a}_2 \\ \cdot & \boldsymbol{a}_{16} & \boldsymbol{a}_{18} & \boldsymbol{a}_{17} & \boldsymbol{a}_{13} & \boldsymbol{a}_9 \\ \cdot & \cdot & \boldsymbol{a}_{21} & \boldsymbol{a}_{20} & \boldsymbol{a}_{15} & \boldsymbol{a}_{11} \\ \cdot & \cdot & \cdot & \boldsymbol{a}_{19} & \boldsymbol{a}_{14} & \boldsymbol{a}_{10} \\ \cdot & \cdot & \cdot & \cdot & \boldsymbol{a}_{12} & \boldsymbol{a}_8 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \boldsymbol{a}_7 \end{pmatrix}$$

is described by 9 independent constants; 3 components by forth order tensor a_1, a_4, a_7 and six independent components of six order tensor $c_1, c_2, c_8, c_{16}, c_{19}, c_{51}$.

In case of isotropic material, among independent constants of forth order tensor the relationship takes place

$$a_4 = \frac{a_1 - a_7}{2},$$

but for elasticity constants of six order tensor tree more relations are obtained

$$c_{16} = \frac{1}{2}(c_2 - c_8),\ c_{19} = \frac{1}{4}(c_1 - c_2)$$
$$c_{51} = \frac{1}{8}(c_1 - 3c_2 + 2c_8).$$

Therefore, the governing equations of second order in case of isotropic materials are expressed from only 5 independent constants.

If interaction among atoms is central, than the following relations exist

$$a_7 = a_4 = \frac{a_1}{3},$$
$$c_8 = c_{16} = c_{51} = \frac{7c_2 - c_1}{6},$$

so, in case of one isotropic material with central interactions, the governing equations of second order are expressed only by tree independent constants.

In case of governing equations of third order may appear the eight order tensors. These tensors are expressed by square matrix of six order, each element represents the six order matrix.

CONCLUSIONS

The possibility of matrix presentation of higher order tensors essentially simplifies the mathematical modeling of nonlinear behavior of anisotropic materials. It was found that the constitutive equations of the second order in the general case of anisotropy are expressed by 77 independent elastic constants.

For cubic symmetry materials the number of independent constants of elasticity is reduced up to 9, (3 independent elastic constants for forth order tensor and 6 independent constants for six order tensor). In case of isotropic materials the number of independent constants of elasticity is reduced up to 5, if interaction between atoms is central, than number of independent constants is reduced up to 3.

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