Left coquotient with respect to join in the class of preradicals in modules

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The purpose of this communication is the definition and investigation of a new inverse operation in the class of preradicals $\mathbb{PR}$ of the category R-Mod of left R-modules.
In [1] and [2] two inverse operations, the left quotient with respect to join and the left coquotient with respect to meet, are defined and studied. These operations exist for any pair of preradicals of $\mathbb{PR}$. In [3] another inverse operation is introduced and investigated, namely the left quotient with respect to meet which, in contrast to the preceding cases, is partial.
In the present lecture in a similar manner as in [3], a new partial inverse operation is defined. We will show the criteria of existence of this operation, its main properties, the relations with the lattice operations of $\mathbb{PR}$ and some particular cases.
We remind that *a preradical* r in the category R-Mod is a subfunctor of identity functor of R-Mod, i.e. $r(M) \subseteq M$ and $f(r(M)) \subseteq r(M')$ for any R-morphism $f : M \to M'$ ([4]).
Definition. Let $r, s \in \mathbb{PR}$. The *left coquotient with respect to join of* r *by* s is defined as the greatest preradical among $r_\alpha \in \mathbb{PR}$ with the property $r_\alpha \# s \leq r$. We denote this preradical by $r \,{}^{\vee}\!/_{\#}\, s$ and we will say that r is the *numerator* and s is the *denominator* of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$.
Lemma 1.(Criteria of existence) Let $r, s \in \mathbb{PR}$. The left coquotient $r \,{}^{\vee}\!/_{\#}\, s$ of r by s with respect to join exists if and only if $r \geq s$ and it can be presented in the form
$r \,{}^{\vee}\!/_{\#}\, s = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r \}$.
Proposition 1. If $r, s \in \mathbb{PR}$, then for any preradical $t \in \mathbb{PR}$ we have:

$$r \geq t \# s \Leftrightarrow r \,{}^{\vee}\!/_{\#}\, s \geq t.$$

Some basic properties of this operation are elucidated. In particular:
1) the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$ is monotone in the numerator and antimonotone in the denominator;
2) $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, t = r \,{}^{\vee}\!/_{\#}\, (t \# s)$; 3) $(r \# s) \,{}^{\vee}\!/_{\#}\, t \geq r \# (s \,{}^{\vee}\!/_{\#}\, t)$;
4) $(r \,{}^{\vee}\!/_{\#}\, t) \,{}^{\vee}\!/_{\#}\, (s \,{}^{\vee}\!/_{\#}\, t) \geq r \,{}^{\vee}\!/_{\#}\, s$; 5) $(r \# t) \,{}^{\vee}\!/_{\#}\, (s \# t) \geq r \,{}^{\vee}\!/_{\#}\, s$;
6) $\left(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha\right) {}^{\vee}\!/_{\#}\, s = \bigwedge_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\vee}\!/_{\#}\, s)$; 7) $\left(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha\right) {}^{\vee}\!/_{\#}\, s \geq \bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\vee}\!/_{\#}\, s)$.

In continuation we consider the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$ in some particular cases ($r = s$, $s = 1$, $r = 0$):
1) $r \,{}^{\vee}\!/_{\#}\, r = c(r)$; 2) $r \,{}^{\vee}\!/_{\#}\, 0 = r$; 3) $1 \,{}^{\vee}\!/_{\#}\, s = 1$,
where $c(r) = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# r = r \}$ is co-equalizer of r ([5]).

The co-equalizer of every preradical r is a radical. It is known that $r \in \mathbb{PR}$ is a radical if and only if $c(r) = r$ ([5]). We have $c(r) \leq r \,{}^{\vee}\!/_{\#}\, s \leq r$ and if r is a radical, then $r \,{}^{\vee}\!/_{\#}\, s = r$.
Further we indicate some properties of co-equalizer relative to the studied operation:

$$c(r) \# (r \,{}^{\vee}\!/_{\#}\, s) = r \,{}^{\vee}\!/_{\#}\, s; \;\; (r \,{}^{\vee}\!/_{\#}\, s) \# c(s) = r \,{}^{\vee}\!/_{\#}\, s; \;\; (r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, c(s) = r \,{}^{\vee}\!/_{\#}\, s.$$

The following statement show the behaviour of $r \,{}^{\vee}\!/_{\#}\, s$ in the case of some special types of preradicals (prime, $\wedge$-prime, irreducible ([6])).

Proposition 2. Let $r, s \in \mathbb{PR}$. Then:
1) If r is prime, then $r \,{}^{\vee}\!/_{\#}\, s$ is prime;
2) If r is $\wedge$-prime, then $r \,{}^{\vee}\!/_{\#}\, s$ is $\wedge$-prime;
3) If r is irreducible and $r = t \# s$ for some preradical $t \in \mathbb{PR}$, then $r \,{}^{\vee}\!/_{\#}\, s$ is irreducible.

The preradicals obtained by this inverse operation are arranged in the following order:

$$r \,{}^{\vee}\!/_{\#}\, s \leq (r \,{}^{\vee}\!/_{\#}\, s) \# s \leq r \leq (r \# s) \,{}^{\vee}\!/_{\#}\, s \leq r \# s.$$

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