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Minimal polynomial basis of $GL(2,\mathbb{R})$-comitants and of $GL(2,\mathbb{R})$-invariants of the planar system of differential equations with nonlinearities of the fourth degree

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Let us consider the system of differential equations with nonlinearities of the fourth degree

$$\frac{dx}{dt} = P_1(x,y) + P_4(x,y), \quad \frac{dy}{dt} = Q_1(x,y) + Q_4(x,y), \tag{1}$$

where P_i and Q_i are homogeneous polynomials of degree i in x and y with real coefficients. We denote by A the 14-dimensional coefficient space of the system (1), by $\mathbf{a} \in A$ the vector of coefficients, by $\boldsymbol{q} \in \mathcal{Q} \subseteq Aff(2,\mathbb{R})$ a nondegenerate linear transformation of the phase plane of the system (1), by $\mathbf{q}$ the transformation matrix and by $r_{\boldsymbol{q}}(\mathbf{a})$ the linear representation of coefficients of the transformed system in the space A.

Definition 1. [1] *A polynomial $\mathcal{K}(\mathbf{a},\mathbf{x})$ in coefficients of system* (1) *and coordinates of the vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ is called a comitant of system* (1) *with respect to the group $\mathcal{Q}$ if there exists a function $\lambda : \mathcal{Q} \to \mathbb{R}$ such that*

$$\mathcal{K}(r_{\boldsymbol{q}}(\mathbf{a}), \mathbf{q}\mathbf{x}) \equiv \lambda(\boldsymbol{q})\mathcal{K}(\mathbf{a},\mathbf{x})$$

for every $\boldsymbol{q} \in \mathcal{Q}$, $\mathbf{a} \in A$ and $\mathbf{x} \in \mathbb{R}^2$.

If $\mathcal{Q}$ is the group $GL(2,\mathbb{R})$, then the comitant is called $GL(2,\mathbb{R})$-comitant or centroaffine comitant. In what follows only $GL(2,\mathbb{R})$-comitants are considered. The function $\lambda(\boldsymbol{q})$ is called a multiplicator. It is known [1] that the function $\lambda(\boldsymbol{q})$ has the form $\lambda(\boldsymbol{q}) = \Delta_{\boldsymbol{q}}^{-g}$, where $\Delta_{\boldsymbol{q}} = \det \boldsymbol{q}$ and g is an integer, which is called the weight of the comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$. If $g = 0$, then the comitant is called absolute, otherwise it is relative.

If a comitant does not depend on the coordinates of the vectors $\mathbf{x}$, then it is called *invariant.*

We say that a comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ has the type $(\rho,\ g,\ l_1,\ l_4)$ if it has the degree ρ with respect to coordinates of the vector $\mathbf{x}$, the weight g and the degree l_i, $i = 1, 4$ with respect to the coefficients of the homogenity of degree i for the system (1).

Definition 2. [4] *Let f and φ be polynomials in the coordinates of the vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ of the degrees r and ρ, respectively. The polynomial*

$$(f,\varphi)^{(k)} = \frac{(r-k)!(\rho-k)!}{r!\rho!} \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k \varphi}{\partial x^h \partial y^{k-h}}$$

is called the transvectant of index k of polynomials f and φ.

If polynomials f and φ are $GL(2,\mathbb{R})$-comitants of system (1), then the transvectant of the index $k \le \min(r,\rho)$ is a $GL(2,\mathbb{R})$-comitant of the system (1). **Definition 3.** [1, 3] *The set*

$\mathcal{S}$ of comitants (respectively, invariants) is called a polynomial basis of comitants (respectively, invariants) for system (1) with respect to a group $\mathcal{Q}$ if any comitant (respectively, invariant) of system (1) with respect to the group $\mathcal{Q}$ can be expressed in the form of a polynomial of elements of the set $\mathcal{S}$.

Definition 4. [3] *The set $\mathcal{S}$ of comitants (respectively, invariants) of degrees less or equal to δ is called a polynomial basis of comitants for the system* (1) *up to the δ degree with respect to the group $\mathcal{Q}$ if any comitant (respectively, invariants) of the degree less or equal to δ of system* (1) *with respect to the group $\mathcal{Q}$ can be expressed as a polynomial of elements of this set $\mathcal{S}$.*

Definition 5. [1, 3] *A polynomial basis of comitants (respectively, invariants) for system (1) with respect to a group $\mathcal{Q}$ is called minimal if by the removal from it of any comitant (respectively, invariant) it ceases to be a polynomial basis.*

The theory of algebraic invariants and comitants for polynomial autonomous systems of differential equations has been developed by C. Sibirschi [1, 2] and his disciples. One of the important problems concerning this theory is the construction of minimal polynomial bases of the invariants and comitants of the mentioned systems, with respect to different subgroups of the affine group of the transformations of their phase planes, in particular with respect to the subgroup $GL(2,\mathbb{R})$. Some important results in this direction are obtained by academician C. Sibirschi [1, 2] and N.Vulpe [3]. We remark, that polynomial bases for different combinations of homogeneous polynomials $P^j_m(x^1,x^2)$ ($j = 1,\ 2,\ m = 0,\ 1,\ 2,\ 3$) in system were considered by E. Gasinskaya-Kirnitskaya, Dang Dinh Bich, D. Boularas, M. Popa, V. Ciobanu, V. Danilyuk, E. Naidenova.

We shall consider the following polynomials:

$$R_i = P_i y - Q_i x; \quad S_i = \frac{1}{i}\left(\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y}\right), \qquad (i = 1, 4), \tag{2}$$

which in fact are $GL(2,\mathbb{R})$-comitants of the first degree with respect to the coefficients of system (1).

Using the comitants (2) as elementary "bricks" and the notion of transvectant we have constructed 419 irreducible $GL(2,\mathbb{R})$-comitants of system (1) and hence, the next results is proved:
Theorem. *A minimal polynomial basis of $GL(2,\mathbb{R})$-comitants (respectively, of $GL(2,\mathbb{R})$-invariants) of system (1) up to 18 degree consists from 419 elements (respectively, 182 elements) which must be of the following 111 (respectively, 42) types:*

$$\begin{array}{llllll}
(2,\ -1,\ 1,\ 0) & -\ 1; & (5,\ -1,\ 0,\ 1) & -\ 1; & (3,\ 0,\ 0,\ 1) & -\ 1;\\
(2,\ 2,\ 0,\ 2) & -\ 3; & (4,\ 1,\ 0,\ 2) & -\ 1; & (6,\ 0,\ 0,\ 2) & -\ 2;\\
(1,\ 1,\ 1,\ 1) & -\ 1; & (3,\ 0,\ 1,\ 1) & -\ 2; & (5,\ -1,\ 1,\ 1) & -\ 1;\\
(0,\ 0,\ 2,\ 0) & -\ 1; & (1,\ 4,\ 0,\ 3) & -\ 2; & (3,\ 3,\ 0,\ 3) & -\ 6;\\
(5,\ 2,\ 0,\ 3) & -\ 3; & (7,\ 1,\ 0,\ 3) & -\ 1; & (9,\ 0,\ 0,\ 3) & -\ 1;\\
(0,\ 3,\ 1,\ 2) & -\ 3; & (2,\ 2,\ 1,\ 2) & -\ 4; & (4,\ 1,\ 1,\ 2) & -\ 3;\\
(6,\ 0,\ 1,\ 2) & -\ 1; & (1,\ 1,\ 2,\ 1) & -\ 2; & (3,\ 0,\ 2,\ 1) & -\ 1;\\
(0,\ 6,\ 0,\ 4) & -\ 6; & (2,\ 5,\ 0,\ 4) & -\ 6; & (4,\ 4,\ 0,\ 4) & -\ 6;
\end{array}$$

(6, 3, 0, 4) – 2;	(1, 4, 1, 3) – 8;	(3, 3, 1, 3) – 6;
(5, 2, 1, 3) – 1;	(7, 1, 1, 3) – 1;	(0, 3, 2, 2) – 1;
(2, 2, 2, 2) – 3;	(4, 1, 2, 2) – 1;	(1, 1, 3, 1) – 1;
(1, 7, 0, 5) – 11;	(3, 6, 0, 5) – 9;	(5, 5, 0, 5) – 1;
(7, 4, 0, 5) – 4;	(0, 6, 1, 4) – 6;	(2, 5, 1, 4) – 9;
(4, 4, 1, 4) – 3;	(1, 4, 2, 3) – 9;	(3, 3, 2, 3) – 1;
(5, 2, 2, 3) – 1;	(0, 3, 3, 2) – 3;	(2, 2, 3, 2) – 2;
(0, 9, 0, 6) – 7;	(2, 8, 0, 6) – 12;	(4, 7, 0, 6) – 2;
(1, 7, 1, 5) – 20;	(3, 6, 1, 5) – 1;	(5, 5, 1, 5) – 1;
(0, 6, 2, 4) – 8;	(2, 5, 2, 4) – 5;	(1, 4, 3, 3) – 4;
(3, 3, 3, 3) – 1;	(0, 3, 4, 2) – 1;	(1, 10, 0, 7) – 20;
(3, 9, 0, 7) – 1;	(5, 8, 0, 7) – 1;	(0, 9, 1, 6) – 15;
(2, 8, 1, 6) – 5;	(1, 7, 2, 5) – 7;	(3, 6, 2, 5) – 1;
(0, 6, 3, 4) – 10;	(1, 4, 4, 3) – 2;	(0, 3, 5, 2) – 1;
(0, 12, 0, 8) – 15;	(2, 11, 0, 8) – 4;	(1, 10, 1, 7) – 7;
(3, 9, 1, 7) – 1;	(0, 9, 2, 6) – 16;	(1, 7, 3, 5) – 2;
(0, 6, 4, 4) – 5;	(1, 4, 5, 3) – 1;	(1, 13, 0, 9) – 5;
(3, 12, 0, 9) – 1;	(0, 12, 1, 8) – 19;	(1, 10, 2, 7) – 2;
(0, 9, 3, 6) – 5;	(1, 7, 4, 5) – 1;	(0, 6, 5, 4) – 3;
(0, 15, 0, 10) – 14;	(1, 13, 1, 9) – 2;	(0, 12, 2, 8) – 5;
(1, 10, 3, 7) – 1;	(0, 9, 4, 6) – 3;	(0, 6, 6, 4) – 1;
(1, 16, 0, 11) – 2;	(0, 15, 1, 10) – 5;	(1, 13, 2, 9) – 1;
(0, 12, 3, 8) – 3;	(0, 9, 5, 6) – 1;	(0, 6, 7, 4) – 1;
(0, 18, 0, 12) – 4;	(1, 16, 1, 11) – 1;	(0, 15, 2, 10) – 3;
(0, 12, 4, 8) – 1;	(0, 9, 6, 6) – 1;	(1, 19, 0, 3) – 1;
(0, 18, 1, 12) – 3;	(0, 15, 3, 10) – 1;	(0, 12, 5, 8) – 1;
(0, 21, 0, 14) – 2;	(0, 18, 2, 12) – 1;	(0, 15, 4, 10) – 1;
(0, 21, 1, 14) – 1;	(0, 18, 3, 12) – 1;	(0, 24, 0, 16) – 1;
(0, 21, 2, 14) – 1;	(0, 24, 1, 16) – 1;	(0, 27, 0, 18) – 1.

Remark. *Besides each of the types given in the theorem above, we have also indicated the number of the irreducible comitants (or invariants) having this type.*

We establishe a conjecture that the minimal polynomial basis of $GL(2,\mathbb{R})$-comitants (respectively, of $GL(2,\mathbb{R})$-invariants) of system (1) consists from 419 elements (respectively, 182 elements) which must be of the above 111 (respectively, 42) types.

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